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## ARRAY OPTIMIZATION CRITERIA

Dr. David K. Cheng

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## **ARRAY OPTIMIZATION CRITERIA**

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**Title of Report : ARRAY OPTIMIZATION CRITERIA**

**FOREWORD**

This report was prepared by Dr. David K. Cheng, Professor of Electrical Engineering, Syracuse University, Syracuse, New York under Contract No. F30602-68-C-0067, ARPA Order No. 1010.

This research was supported by the Advanced Research Projects Agency of the Department of Defense and was monitored by RADC Project Engineer Joseph LoVecchio (EMATS).

Information in this report is embargoed under the U. S. Export Control Act of 1949, administered by the Department of Commerce.

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## ABSTRACT

This report summarizes the major technical accomplishments during the period from 22 September 1967 to 21 September 1968 under Contract No. F30602-68-C-0067 (ARPA Order No. 1010). The accomplishments can be grouped under three headings; namely, (1) development of a new method for directivity or signal-to-noise optimization by spacing perturbation, (2) development of a new technique for the optimization of arrays with a large number of elements, and (3) development of a new method for pattern synthesis of circular arrays with directive elements. Essential formulations as well as analytical justifications are outlined.

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## I. INTRODUCTION

"Array Optimization Criteria" was the title of Contract No. F30602-68-C-0067 sponsored by the Advanced Research Projects Agency under ARPA Order No. 1010 and monitored by the Rome Air Development Center of the Air Force System Command. This is a technical report summarizing the accomplishments under the contract for the period from 22 September 1967 to 21 September 1968.

The accomplishments can be grouped under three major tasks. They are:

- (1) Development of a new method for directivity or signal-to-noise optimization by spacing perturbation.
- (2) Development of a new technique for the optimization of large arrays.
- (3) Development of a new method for pattern synthesis of circular arrays with directive elements.

The analytical techniques developed for the above tasks are outlined in the following sections.



## II. SPACING PERTURBATION TECHNIQUES FOR ARRAY OPTIMIZATION<sup>1</sup>

Until the development of the technique to be outlined in this section, array optimization usually starts with an array of a given configuration, and maximization of a performance index is achieved by properly adjusting the excitation amplitudes and phases in the array elements. However, for a given set of amplitude and phase values, uniform spacing does not yield the highest obtainable directivity or signal-to-noise ratio. The optimum element spacings can be found by a spacing perturbation technique. The basis of the technique lies in an optimization theorem which will be proved in the Appendix. The starting point can be an array of arbitrary (uniform or nonuniform) spacings with nonoptimum excitation amplitudes and phases. After an optimum set of spacings is obtained, the excitation amplitudes and phase shifts can be adjusted for further improvement in the desired performance index; then the optimum spacings can be recalculated for the new excitation and the cycle repeated if desired. In particular, the technique provides a method for improving the directivity or the output signal-to-noise ratio of an array with any given excitation and spacings by spacing perturbation until a maximum is obtained.

Consider a linear array of  $2N + \eta$  identical antenna elements symmetrically located about the origin, with  $\eta = 1$  when the total number of elements is odd, and  $\eta = 0$  when the total number of elements is even and the center element is absent. Let  $\theta_0$  be the angle which the direction of the signal makes with the array axis. The excitation in the  $m$ th element from the origin is  $I_m \exp(j\phi_m)$ , where

$$\phi_m = - (2\pi d_m / \lambda) \cos \theta_o - \phi_m \quad . \quad (1)$$

In (1),  $d_m$  is the distance of the  $m$ th element from the origin,  $\lambda$  is the signal wavelength, and  $\phi_m$  is the phase shift from the cophasal operation. The array factor will then be

$$E(\psi) = \frac{I_o}{2} + \sum_{m=1}^N I_m \cos (D_m \psi - \phi_m) , \quad (2)$$

where

$$\psi = (2\pi d / \lambda) (\cos \theta - \cos \theta_o) \quad (3)$$

$$D_m = d_m / d \quad (4)$$

and  $d$ , a normalizing distance, may be any choice of convenience. For example, if one starts with a uniformly spaced array, it would be natural to make  $d$  the spacing between neighboring elements. We define the output signal-to-noise ratio as the ratio of the power received per unit solid angle in the direction of the signal to the average noise power received per unit solid angle. Thus,

$$G_{\text{SNR}} = \frac{|E(o)|^2}{\frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi |E(\psi)|^2 w(\theta, \phi) \sin \theta d\theta}$$

where

$$w(\theta, \phi) = g(\theta, \phi) T(\theta, \phi) \quad (6)$$

$$g(\theta, \phi) = \text{element power pattern function} \quad (7)$$

$$T(\theta, \phi) = \text{spatial distribution function of noise power} \quad (8)$$

It is convenient to normalize  $g(\theta, \phi)$  in (7) with respect to its value in the direction of the signal; i.e.,  $g(\theta_o, \phi_o) = 1$ . The composite function  $w(\theta, \phi)$  in (6) can be viewed as a weighting function on the array power pattern  $|E(\psi)|^2$ . We note that (5) becomes the expression for directivity,  $G_o$ , when  $T(\theta, \phi) = 1$ ; hence it serves as the starting point for the optimization of both  $G_{\text{SNR}}$  and  $G_o$ . Clearly, both are affected by the normalized element positions  $\{D_m\}$  and the excitations  $\{I_m, \phi_m\}$ .

Let the perturbed normalized element positions be  $\{D_m\}$ .

$$D_m = D_m^o + x_m, \quad (9)$$

where  $D_m^o$  and  $x_m$  represent, respectively, the unperturbed position and the spacing perturbation for the  $m$ th element and  $x_m \ll 1$ . Substitution of (9) in (2) yields approximately

$$E(\psi) = E^o(\psi) - \sum_{m=1}^N x_m [I_m \psi \sin(D_m^o \psi - \phi_m)], \quad (10)$$

where  $E^o(\psi)$  is the unperturbed array factor with  $D_m^o$  substituted for  $D_m$  in (2). Using (10), we can write (5) in the following form:

$$G_{\text{SNR}} = \frac{|E^o(0)|^2}{A - 2\bar{x}' \bar{\beta} + \bar{x}' \bar{C} \bar{x}} \quad (11)$$

where

$$A = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi |E^o(\psi)|^2 w(\theta, \phi) \sin \theta d\theta \quad (12)$$

$$\bar{x}' = [x_1, x_2, \dots, x_m, \dots, x_N] \quad (13)$$

is the transpose of the column matrix of spacing perturbations  $\bar{x}$ ;  $\bar{\beta}$  is a column matrix of typical element

$$\beta_m = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi I_m \psi E^0(\psi) w(\theta, \phi) \sin(D_m^0 \psi - \phi_m) \sin \theta d\theta ; \quad (14)$$

and  $\bar{C} = [c_{mk}]$  is an  $N \times N$  square matrix with

$$c_{mk} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi I_m I_k \psi^2 w(\theta, \phi) \sin(D_m^0 \psi - \phi_m) \sin(D_k^0 \psi - \phi_k) \sin \theta d\theta \quad (15)$$

It can readily be shown that  $\bar{C}$  is symmetric and positive definite.

Use can then be made of the theorem proved in the Appendix, which enables us to conclude:

$$\text{a) } \begin{aligned} \text{Max. } G_{\text{SNR}} &= \frac{|E^0(0)|^2}{A - \bar{\beta}' \bar{C}^{-1} \bar{\beta}} \\ (\bar{x} = \bar{x}_M) \end{aligned} \quad (16)$$

$$\text{b) } \bar{x}_M = \bar{C}^{-1} \bar{\beta} \quad (17)$$

Equations (16) and (17) give the results of a first-order perturbation.

After the components of  $\bar{x}_M$  have been determined from (17), they can be substituted back in (9). One can then use  $(\bar{D}^0 + \bar{x}_M)$  as the new normalized element-position column matrix and perform a second-order perturbation to obtain further improvement in the performance index. This process can be repeated until it becomes evident that further iteration yields a negligible improvement. For the many cases which have been computed, it is found that convergence toward optimum values usually takes place very quickly; seldom are more than two iterations required.

The spacing perturbation technique outlined above yields the required element positions in order to maximize  $G_{\text{SNR}}$  for a given set of excitation parameters. Even if one starts with a uniformly spaced array, the element spacings will no longer be uniform after the perturbation. Now this perturbed, nonuniformly spaced array can be further optimized by proper amplifications and phase shifts following the array elements. With this in mind, we let  $y_0 = I_0$ ,  $y_n = I_n \cos \phi_n$ , and  $y_{N+n} = I_n \sin \phi_n$  ( $n=1,2,\dots,N$ ). Furthermore, let  $h_0(\psi) = \eta/2$ ,  $h_n(\psi) = \cos D_n \psi$ , and  $h_{N+n}(\psi) = \sin D_n \psi$  ( $n=1,2,\dots,N$ ). Equation (5) can then be converted to the following form:

$$G_{\text{SNR}} = \frac{\bar{y}' \bar{h}_0 \bar{h}_0' \bar{y}}{\bar{y}' \bar{B} \bar{y}}, \quad (18)$$

where

$$\bar{y}' = [y_0, y_1, y_2, \dots, y_{2N}] \quad (19)$$

$$\bar{h}_0' = [h_0(0), h_1(0), h_2(0), \dots, h_{2N}(0)] \quad (20)$$

and  $\bar{B} = [b_{ij}]$  is a  $(2N + n) \times (2N + n)$  square matrix with

$$b_{ij} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi h_i(\psi) h_j(\psi) w(\theta, \phi) \sin \theta d\theta. \quad (21)$$

$G_{\text{SNR}}$  in (18) is expressed as a ratio of two quadratic forms. Since  $\bar{B}$  is symmetric and positive definite, we know<sup>2,3</sup> that  $G_{\text{SNR}}$  can be maximized by choosing the column matrix  $\bar{y} = \bar{y}_M$ :

$$\bar{y}_M = \bar{B}^{-1} \bar{h}_0 \quad (22)$$

and that

$$\begin{aligned} \text{Max. } G_{\text{SNR}} &= \bar{\mathbf{h}}_0' \bar{\mathbf{B}}^{-1} \bar{\mathbf{h}}_0 \\ (\bar{\mathbf{y}} &= \bar{\mathbf{y}}_M) \end{aligned} \quad (23)$$

Equation (22) completely specifies the amplifications and phase shifts required to obtain the maximum possible  $G_{\text{SNR}}$ , as given in (23), for the perturbed array. If the performance index of interest is directivity  $G_0$ ,  $\bar{\mathbf{y}}_M$  will yield the required excitation amplitudes and phases to maximize  $G_0$ . We have now reached a second submaximum, which may possibly be further improved by holding the excitation unchanged and again perturbing the spacings. The cycle may be repeated until further adjustments are no longer worthwhile.

It is to be emphasized that this alternate spacing perturbation and excitation adjustment procedure of seeking a maximum performance index can be applied to an array which initially has an arbitrary, nonuniform spacing and an arbitrary distribution of excitation amplitudes and phases. Optimization using the above technique has been carried out for both broadside and endfire arrays for typical spatial distributions of noise or clutter power.<sup>1</sup> For example, with a cosine-square over a pedestal type of clutter power distribution, the maximum signal-to-noise ratio for a 7-element broadside uniform array is 16.0 and that for an optimized array is increased to 181.9. Even more impressive improvement is obtained for a 7-element endfire array, in which case the signal-to-noise ratio is increased from 15.8 to 891.4 by optimization. Moreover, the improvement does not result in the objectionable features (very small element spacings or large excitation amplitudes of alternating signs) of a supergain array.

### III. METHODS FOR OPTIMIZATION OF LARGE ARRAYS

Methods for the optimization of either the directivity or the signal-to-noise ratio of an N-element antenna array requires the inversion of an  $N \times N$  square matrix. This requirement raises questions on the achievable accuracy and the amount of computer time necessary for handling arrays with a large number of elements. For large arrays, it would be extremely desirable if a technique could be found so that the computational difficulties would be reduced in the use of the optimization method based on a consideration of the ratio of two Hermitian forms. Such a technique has been found and will be outlined in this section.

The basis of this technique lies in a transformation which transforms the excitations in the elements of an array into finite summations of an exponential series with complex coefficients. Let the normalized array factor of an N-element array be written as

$$E_n(\psi) = \frac{1}{N} \sum_{m=1}^N I_m \exp[j (N - 2m + 1) \psi] , \quad (24)$$

where  $\psi$  has been defined in (3), and N is very large. We set

$$I_m = \sum_{p=1}^N a_p \exp[-j (p-1)(N - 2m + 1) \pi/N] . \quad (25)$$

It is easy to verify that the substitution of (25) in (24) yields

$$E_n(\psi) = \frac{1}{N} \sum_{p=1}^N a_p f_p(\psi) \quad (26)$$

$$f_p(\psi) = \frac{\sin \frac{N}{2} [\psi - (p-1) 2\pi/N]}{\sin \frac{1}{2} [\psi - (p-1) 2\pi/N]} . \quad (27)$$

Now the expression in (27) has the following property:

$$f_p(\psi_{p'}) = N\delta_{pp'} , \quad (28)$$

where

$$\psi_{p'} = \frac{2}{N} (p' - 1) \quad (29)$$

and  $\delta_{pp'}$  is the Kronecker delta. Combination of (27), (28) and (29) gives a very simple result:

$$E_n(\psi_p) = a_p , \quad (30)$$

which says that the sampled value of the normalized array factor in the direction specified by  $\psi_p$  is numerically equal to the transformed coefficient  $a_p$  in (25). Since  $a_p$  decreases rapidly as  $p$  increases, only very few terms need be used for the summation in (26). It is noted that the inverse relation for (25) is

$$a_p = \frac{1}{N} \sum_{m=1}^N I_m \exp[j (p-1)(N - 2m + 1) \pi/N] . \quad (31)$$

For a properly chosen positive integer  $P \ll N$ , we have  $|a_p/a_1| \ll 1$  for  $P \leq p \leq N-P$ , and

$$E_n(\psi) \approx \frac{1}{N} \sum_{p=1}^{2P} b_p g_p(\psi) , \quad (32)$$

where

$$b_p = a_p , \quad g_p(\psi) = f_p(\psi) \text{ for } 1 \leq p \leq P \quad (33)$$



and

$$b_p = a_{N-2P+p}, g_p(\psi) = f_{N-2P+p}(\psi) \text{ for } P+1 \leq p \leq 2P. \quad (34)$$

The element excitations given in (25) becomes

$$I_m = \sum_{p=1}^P b_p \exp[-j(p-1)(N-2m+1)\pi/N] \\ + \sum_{p=P+1}^{2P} b_p \exp[-j(N-2P+p-1)(N-2m+1)\pi/N], \quad (35)$$

This approximate expression for  $I_m$  can then be used in the formulation for the array performance index of interest, and the computation labor for its maximization is greatly reduced.

Preliminary investigation has indicated that 8 values of  $a_p$  give a very good approximation of the array pattern  $E_n(\psi)$  for a 50-element linear array. This means that the tedious and inaccurate inversion of a  $50 \times 50$  matrix can be replaced by the inversion of an  $8 \times 8$  matrix through the use of transformation (25). It remains to obtain more numerical data for arrays of different geometrical configurations and for different scan angles. Study will also be made on the effect of the approximation on meaningful constraints such as main-beam radiation efficiency.

#### IV. PATTERN SYNTHESIS FOR CIRCULAR ARRAYS WITH DIRECTIVE ELEMENTS

Circular arrays consisting of radiating sources uniformly disposed around a circular ring find applications in situations where steerability and pattern invariance are of importance. Several authors<sup>4-7</sup> have dealt with pattern synthesis techniques for circular arrays. Most published articles pay special attention to patterns with equal sidelobes and none appears to have considered nonisotropic radiating elements. Unlike a linear array, a straightforward multiplication of the element pattern function and the array factor does not correctly give the radiation pattern of a circular array. This causes complication in pattern analysis and difficulty in pattern synthesis. It should be pointed out that the problem of pattern synthesis is different from that of gain optimization, the latter problem having been solved previously for circular arrays with directive elements.<sup>8</sup>

Recently Mott and Dudgeon<sup>9</sup> proposed a method for synthesizing the pattern of a circular array with isotropic sources. It involves a preliminary integration, leading to a matrix which can be reduced and inverted on a digital computer with less labor. However, when the number of array elements is large, error accumulation and storage problems in the reduction and inversion of a large matrix would again become serious. In this section we present a refined method which does not require a matrix inversion at all and hence is particularly useful when large circular arrays with many elements are to be designed. Moreover, the present method is applicable to cases where the array elements themselves possess a directive pattern.

Consider a circular array of radius  $\underline{a}$  with  $M + 1$  directive elements arranged uniformly around its periphery. If  $g(\phi)$  denotes the element pattern, the far-field pattern of the array in the plane of the circle is

$$E(\phi) = \sum_{m=0}^M A_m g\left(\phi - \frac{2m\pi}{M+1}\right) \exp[j\beta a \cos\left(\phi - \frac{2m\pi}{M+1}\right)] , \quad (36)$$

where  $\beta = 2\pi/\lambda$  and  $A_m$  is the complex excitation in the  $m$ th element. In (36), the zeroth element is assumed to lie on the reference axis from which the angle  $\phi$  is measured. Let  $E_o(\phi)$  be the radiation pattern to be synthesized. Since  $E_o(\phi)$  is periodic in  $\phi$ , it can be expanded in a Fourier-series form:

$$E_o(\phi) = \sum_{n=-N}^N B_n \exp(jn\phi) , \quad (37)$$

where

$$B_n = \frac{1}{2\pi} \int_0^{2\pi} E_o(\phi) \exp(-jn\phi) d\phi . \quad (38)$$

Equation (27) is a truncated Fourier series of  $2N + 1$  terms; it represents a least-mean-square approximation. This approximation is necessary and reasonable because one can not use an infinite number of elements in a practical design and because the avoidance of supergain phenomenon constrains the minimum allowable element spacing. The synthesis problem is then the determination of the excitation coefficients  $\{A_m\}$  such that  $E(\phi)$  in (36) equals  $E_o(\phi)$  in (37). We write

$$\sum_{m=0}^M A_m g\left(\phi - \frac{2m\pi}{M+1}\right) \exp[j\beta a \cos\left(\phi - \frac{2m\pi}{M+1}\right)] = \sum_{n=-N}^N B_n \exp(jn\phi) , \quad (39)$$

for  $0 \leq \phi \leq 2\pi$  .

For a unique solution of (39), we must have  $M + 1 = 2N + 1$ , or  $M = 2N$ . Integrating both sides of (39) in the manner of (38), we obtain

$$\sum_{m=0}^M A_m \alpha_n \exp[-j2mn\pi/(M+1)] = B_n , \quad (40)$$

where

$$\alpha_n = \frac{1}{2\pi} \int_0^{2\pi} g(\phi) \exp[j(\beta a \cos \phi - n\phi)] d\phi , \quad (41)$$

$n = 0, \pm 1, \pm 2, \dots, \pm N$ .

With the reasonable assumption that the element pattern  $g(\phi)$  is an even function of  $\phi$ , we can write

$$g(\phi) = \sum_{k=0}^K C_k \cos k\phi . \quad (42)$$

Substitution of (42) in (41) yields

$$\alpha_n = \frac{1}{2} \sum_{k=0}^K C_k [j^{n-k} J_{n-k}(\beta a) + j^{n+k} J_{n+k}(\beta a)] , \quad (43)$$

$n = 0, \pm 1, \pm 2, \dots, \pm N$  .

Let  $\{C_k\}$  be such that  $\alpha_n \neq 0$  for all  $n$ . Then, by using the following identity,

$$\sum_{n=-N}^N \exp[j2(m' - m)n\pi/(2N + 1)] = (2N + 1)\delta_{mm'} , \quad (44)$$

where  $\delta_{mm'}$  is the Kronecker delta (which equals zero when  $m \neq m'$  and equals unity when  $m = m'$ ), we obtain, from (40),

$$A_m = \frac{1}{2N + 1} \sum_{n=-N}^N \frac{B_n}{\alpha_n} \exp[j 2mn\pi/(2N + 1)] , \quad (45)$$

$$m = 0, 1, 2, \dots, M(\text{or } 2N).$$

We note that (45) is an explicit expression for finding the excitation coefficients,  $\{A_m\}$ , in the  $M + 1$  or  $2N + 1$  elements of the circular array. The  $\{A_m\}$  so determined will yield a radiation pattern which approximates the desired  $E_o(\phi)$  in the least-mean-square sense. The usefulness of (45) stems not only from its applicability to the synthesis of circular arrays with directive elements, but also from the fact that no matrix reduction or inversion is required. This latter feature is a most desirable one when large circular arrays with many elements are to be synthesized.

## V. APPENDIX

In this section the theorem which forms the basis of the spacing perturbation technique described in Section II is stated and proved.

Theorem. If a quantity  $P$  can be expressed in terms of an  $N \times 1$  real column vector  $\bar{x}$  as

$$P = A - 2\bar{x}'\bar{\beta} + \bar{x}'\bar{C}\bar{x}, \quad (46)$$

where  $A$  is a constant,  $\bar{\beta}$  is another  $N \times 1$  real column vector,  $\bar{x}'$  is the transpose of  $\bar{x}$ , and  $\bar{C}$  is an  $N \times N$  positive definite, symmetric, square matrix, then

$$a) \quad \min_{(\bar{x} = \bar{x}_M)} P = A - \bar{\beta}'\bar{C}^{-1}\bar{\beta}, \quad \text{and} \quad (47)$$

$$b) \quad \bar{x}_M = \bar{C}^{-1}\bar{\beta}. \quad (48)$$

Proof: If  $\bar{C}$  is positive definite, it is known that<sup>10</sup>

$$(\bar{\beta}'\bar{C}^{-1}\bar{\beta})(\bar{x}'\bar{C}\bar{x}) \geq (\bar{x}'\bar{\beta})^2 \quad (49)$$

or

$$\bar{x}'\bar{C}\bar{x} \geq \frac{1}{\bar{\beta}'\bar{C}^{-1}\bar{\beta}} (\bar{x}'\bar{\beta})^2, \quad (50)$$

where the equality sign applies when

$$\bar{x} = \bar{x}_M = \bar{C}^{-1}\bar{\beta}. \quad (51)$$

Let  $c = \bar{\beta}'\bar{C}^{-1}\bar{\beta} > 0$ , and  $b = \bar{x}'\bar{\beta}$ . We have, from (46) and (50),

$$\begin{aligned} P &= A - 2b + \bar{x}'\bar{C}\bar{x} \\ &\geq A - 2b + \frac{b^2}{c} \end{aligned} \quad (52)$$

But,

$$\begin{aligned} A - 2b + \frac{b^2}{c} &= A - c + \frac{1}{c}(c-b)^2 \\ &\geq A - c. \end{aligned} \quad (53)$$

Combining (52) and (53), we obtain

$$P \geq A - \bar{\beta}' \bar{C}^{-1} \bar{\beta} \quad , \quad (54)$$

where the equality sign holds with (51); hence the theorem is proved.

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## 13. ABSTRACT

This report summarizes the major technical accomplishments during the period from 22 September 1967 to 21 September 1968, under ~~Contract No. F30602-68-C-0067 (ARPA Order No. 1010)~~. The accomplishments can be grouped under three headings; namely, (1) development of a new method for directivity or signal-to-noise optimization by spacing perturbation, (2) development of a new technique for the optimization of arrays with a large number of elements, and (3) development of a new method for pattern synthesis of circular arrays with directive elements. Essential formulations as well as analytical justifications are outlined.

14.

## KEY WORDS

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Circular Array Synthesis